Solution 3 by Arkady Alt, San Jose, California, USA. Let $a = BC = CA = AB, x = BX, y = CY, z = AZ, u = PX, v = PY, w = PZ$ and $h = \frac{a\sqrt{3}}{2}$ be height of the equilateral triangle ABC, then

$$[ABC] = [PBC] + [PCA] + [PAB] \Leftrightarrow \frac{ah}{2} = \frac{au}{2} + \frac{av}{2} + \frac{az}{2} \iff u + v + w = h$$

and

$$\frac{BX + CY + AZ}{PX + PY + PZ} = \frac{x + y + z}{u + v + w} = \frac{x + y + z}{h} = \frac{2\,(x + y + z)}{a\sqrt{3}}$$

Applying Pythagorean theorem to chain of right triangles $\triangle PXB, \triangle PBZ, \triangle PZA, \triangle PAY,$ $\triangle PXB, \triangle PYC, \triangle PCX$, we obtain

$$\begin{cases} u^2 + x^2 = w^2 + (a - z)^2 \\ w^2 + z^2 = v^2 + (a - y)^2 \\ v^2 + y^2 = u^2 + (a - x)^2 \end{cases}$$

Adding all equations we get

$$\sum_{cyc} \left(u^2 + x^2\right) = \sum_{cyc} \left(w^2 + (a - z)^2\right) \Leftrightarrow 3a^2 = 2a\,(x + y + z)$$

S0, $x + y + z = \frac{3a}{2}$ and, therefore, $\frac{BX + CY + AZ}{PX + PY + PZ} = \sqrt{3}$. □

Also solved by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.

63. *How many ways are there to weigh of* 31 *grams with a balance if we have* 7 *weighs of one gram,* 5 *of two grams, and* 6 *of five grams, respectively?*

(Training Catalonian Team for OME 2014)

Solution 1 by José Luis Díaz-Barrero BARCELONA TECH, Barcelona, Spain. The required number is the number of solutions of $x + y + z = 31$ with

$$x \in \{0, 1, 2, 3, 4, 5, 6, 7\}, \quad y \in \{0, 2, 4, 6, 8, 10\}, \quad z \in \{0, 5, 10, 15, 20, 25, 30\}$$

We claim that the number of solutions of this equation equals the coefficient of x^{31} in the product

$$(1 + x + x^2 + \ldots + x^7)\,(1 + x^2 + x^4 + \ldots + x^{10})\,(1 + x^5 + x^{10} + \ldots + x^{30})$$

Indeed, a term with x^{31} is obtained by taking some term x from the first parentheses, some term y from the second, and z from the third, in such a way that $x+y+z = 31$. Each such possible selection of x, y and z contributes 1 to the considered coefficient of x^{31} in the product. Since,

$$(1 + x + x^2 + \ldots + x^7)\,(1 + x^2 + x^4 + \ldots + x^{10})\,(1 + x^5 + x^{10} + \ldots + x^{30})$$
$$= 1 + x + \ldots + 10x^{30} + 10x^{31} + 10x^{32} + \ldots + x^{46} + x^{47},$$

then the number of ways to obtain 31 grams is 10, and we are done. □

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. We are looking for the number of triplets (k, l, m) such that

$$k \in \{0, \ldots, 7\},\ l \in \{0, \ldots, 5\},\ m \in \{0, \ldots, 6\},\ k + 2l + 5m = 31.$$

Since $k+2l \leq 17$ we conclude that $5m \geq 14$, so $m \in \{3,4,5,6\}$.

- $m=3$, then $k+2l=16$ or $k=2(8-l) \geq 2(8-5)=6$ this gives the unique solution $(k,l,m)=(6,5,3)$.
- $m=4$, then $k+2l=11$ or $k-1=2(5-l) \leq 6$. So every $l \in \{2,3,4,5\}$ yields a solution, and we get four solutions:
$$(k,l,m) \in \{(7,2,4),\ (5,3,4),\ (3,4,4),\ (1,5,4)\}$$
- $m=5$, then $k+2l=6$ or $k=2(3-l)$. So every $l \in \{0,1,2,3\}$ yields a solution, and we get four solutions:
$$(k,l,m) \in \{(6,0,5),\ (4,1,5),\ (2,2,5),\ (0,3,5)\}$$
- $m=6$, then $k+2l=1$, and this yields the unique solution $(k,l,m)=(1,0,6)$.

So, the total number of ways to weight 31 grams is 10. □

Solution 3 by Arkady Alt, San Jose, California, USA. We have to compute the number of elements of the set

$$S := \{(x,y,z) \mid x,y,z \in \mathbb{Z} \text{ and } x+2y+5z=31, 0 \leq x \leq 7, 0 \leq y \leq 5, 0 \leq z \leq 6\}$$
$$= \{(31-2y-5z,y,z) \mid y,z \in \mathbb{Z} \text{ and } 24 \leq 2y+5z \leq 31, 0 \leq y \leq 5, 0 \leq z \leq 6\}.$$

Since in integers $24 \leq 2y+5z \leq 31 \iff 24-2y \leq 5z \leq 31-2y \iff \left\lceil \frac{28-2y}{5} \right\rceil \leq z \leq \left[\frac{31-2y}{5} \right]$ and for any $0 \leq y \leq 5$ holds $\left[\frac{31-2y}{5} \right] \leq 6, \left\lceil \frac{28-2y}{5} \right\rceil > 0$ then, denoting $t := 5-y$, we obtain $\left[\frac{31-2y}{5} \right] = \left[\frac{21+2t}{5} \right] = 4 + \left[\frac{2t+1}{5} \right]$, $\left\lceil \frac{28-2y}{5} \right\rceil = \left\lceil \frac{18+2t}{5} \right\rceil = 3 + \left\lceil \frac{2t+3}{5} \right\rceil$ and

$$S = \left\{ (31-2y-5z, 5-t, z) \mid t,z \in \mathbb{Z} \text{ and } 0 \leq t \leq 5, 3 + \left\lceil \frac{2t+3}{5} \right\rceil \leq z \leq 4 + \left[\frac{2t+1}{5} \right] \right\}.$$

Hence, $|S| = \sum_{t=0}^{5} \left(2 + \left[\frac{2t+3}{5} \right] - \left[\frac{2t+1}{5} \right] \right) = 12 + \sum_{t=0}^{5} \left(\left[\frac{2t+1}{5} \right] - \left[\frac{2t+3}{5} \right] \right)$.

Noting that $\sum_{t=0}^{5} \left(\left[\frac{2t+1}{5} \right] - \left[\frac{2t+3}{5} \right] \right) = \sum_{t=0}^{5} \left[\frac{2t+1}{5} \right] - \sum_{t=1}^{6} \left[\frac{2t+1}{5} \right] = \left[\frac{2 \cdot 0+1}{5} \right] - \left[\frac{2 \cdot 6+1}{5} \right] = -\left[\frac{13}{5} \right] = -2$ we get $|S| = 12-2 = 10$. □

Also solved by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.

64. *Let $A(x)$ be a polynomial with integer coefficients such that for $1 \leq k \leq n+1$, holds:*

$$A(k) = 5^k$$

Find the value of $A(n+2)$.

(Training UPC Team for IMC 2014)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.